Section 4.1: Function, Domain, and Image

Let X and Y be sets.

1. A **function** f from X to Y is a rule which assigns one and only one element $f(x) \in Y$ to each element $x \in X$.
2. We denote a function f from X to Y by $f : X \to Y$.
3. The set X is called the **domain** of f. We write $X = \mathrm{dom}(f)$.
4. The subset $\{y \in Y \mid y = f(x) \text{ for some } x \in X\}$ of Y is called the **range** or the **image** of f. We write the image of f as $\mathrm{im}(f)$.

Surjective, Injective, and Bijective

Let X and Y be sets, and let $f : X \to Y$ be a function.

1. f is called **surjective** if $\mathrm{im}(f) = Y$.
2. f is called **injective** if $x_1, x_2 \in X$ and $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$. Thus, f is injective if distinct inputs have distinct outputs.
3. The statement that f is injective is equivalent to

$$\text{for all } x_1, x_2 \in X, f(x_1) = f(x_2) \text{ implies that } x_1 = x_2.$$

4. f is called **bijective** if it is both injective and surjective.

Identity, Composition, Inverse, and Invertible

Let X, Y and Z be sets.

1. The **identity function** $\mathrm{id}_X : X \to X$ is the function defined by $\mathrm{id}_X(x) = x$ for all $x \in X$.
2. Given functions $f : X \to Y$ and $g : Y \to Z$, the **composition** $g \circ f$ is the function $g \circ f : X \to Z$ defined by $g \circ f(x) = g(f(x))$ for all $x \in X$.
3. Given a function $f : X \to Y$, an **inverse function** for f is a function $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.
4. If a function f has an inverse function f^{-1}, we say that f is an **invertible** function.

Theorem 4.3: A function is invertible if and only if it is bijective.

Linear Transformations

- Let $f : V \to W$ be a function from a vector space V to a vector space W. Then f is called a **linear transformation** if the following two conditions are satisfied:
  1. For all $\mathbf{x}, \mathbf{y} \in V$, we have $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$.
  2. For all $\mathbf{x} \in V$ and $\alpha \in \mathbb{R}$, we have $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$.

- **Theorem 4.7:** Let A be an $m \times n$ matrix. Then A defines a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

  Moreover, given any linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$, there is an $m \times n$ matrix A such that $f(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for all $\vec{\mathbf{x}} \in \mathbb{R}^n$.